at significance level $0.05 = 1 - 0.95$.

6.3.15

(a) To check unbiasedness, the expectation must be computed. $E_\theta(x_1) = 1 \cdot \theta + 0 \cdot (1-\theta) = \theta$. Hence, x_1 is an unbiased estimator of θ.
(b) Since the value of x_1 is only 0 or 1, the equation $x_1^2 = x_1$ always holds. Thus, $E_\theta(x_1^2) = E_\theta(x_1) = \theta$. Hence, x_1^2 is not an unbiased estimator of θ^2. In this exercise, we showed an unbiased estimator is not transformation invariant.

6.3.23

(a) First of all, $(n-1)s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n [x_i^2 - 2\bar{x}x_i + \bar{x}^2] = \sum_{i=1}^n x_i^2 - n\bar{x}^2$. The expectation of the first summation term is

$$E\Big[\sum_{i=1}^n x_i^2\Big] = nE[X^{1/2}] = n(\mu^2 + \sigma^2).$$

Since $n\bar{x}^2 = n^{-1}\sum_{i=1}^n x_i \sum_{j=1}^n x_j$,

$$E[n\bar{x}^2] = \frac{1}{n}E\Big[\sum_{i=1}^n\sum_{j=1}^n x_i x_j\Big] = \frac{1}{n}\sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^n E[x_i x_j] + \frac{1}{n}\sum_{i=1}^n E[x_i]$$
$$= \frac{1}{n}\cdot n(n-1)\cdot\mu^2 + \frac{1}{n}\cdot n\cdot(\mu^2+\sigma^2) = n\mu^2 + \sigma^2.$$

Hence, $E[(n-1)s^2] = n(\mu^2+\sigma^2) - (n\mu^2+\sigma^2) = (n-1)\sigma^2$. Therefore $E[s^2] = \sigma^2$ and s^2 is an unbiased estimator of the variance σ^2.
(b) Let $\hat{\sigma}^2 = (n-1)s^2/n$. The bias of $\hat{\sigma}^2$ is

$$\text{bias}(\hat{\sigma}^2) = E[\hat{\sigma}^2] - \sigma^2 = ((n-1)/n)E[s^2] - \sigma^2 = [(n-1)/n]\sigma^2 - \sigma^2$$
$$= -\sigma^2/n.$$

Hence, the bias $-\sigma^2/n$ converges to 0 as $n \to \infty$.

6.3.24

(a) Since T_1 and T_2 are unbiased estimators of $\psi(\theta)$, $E[T_1] = E[T_2] = \psi(\theta)$. Hence, $E[\alpha T_1+(1-\alpha)T_2] = \alpha E[T_1]+(1-\alpha)E[T_2] = \alpha\psi(\theta)+(1-\alpha)\psi(\theta) = \psi(\theta)$. Therefore, $\alpha T_1 + (1-\alpha)T_2$ is also an unbiased estimator of $\psi(\theta)$.
(b) From Theorem 3.3.4, 3.3.1 (b) and 3.3.2, $\text{Var}_\theta(\alpha T_1+(1-\alpha)T_2) = \text{Var}_\theta(\alpha T_1)+ Var_\theta((1-\alpha)T_2)+2\text{Cov}_\theta(\alpha T_1,(1-\alpha)T_2) = \alpha_\theta^2\text{Var}(T_1)+(1-\alpha)_\theta^2\text{Var}_\theta(T_2)+2\alpha(1-\alpha)\text{Cov}_\theta(T_1,T_2)$. The independence between T_1 and T_2 implies $\text{Cov}_\theta(T_1,T_2) = 0$ and $\text{Var}_\theta(\alpha T_1 + (1-\alpha)T_2) = \alpha_\theta^2\text{Var}(T_1) + (1-\alpha)_\theta^2\text{Var}(T_2)$. (c) The variance of $\alpha T_1 + (1-\alpha)T_2$ can be written as

$$\begin{aligned}&\alpha^2(\text{Var}_\theta(T_1) + \text{Var}_\theta(T_2)) - 2\alpha\text{Var}_\theta(T_2) + \text{Var}_\theta(T_2)\\ &= (\text{Var}_\theta(T_1) + \text{Var}_\theta(T_2))\left(\alpha - \frac{\text{Var}_\theta(T_2)}{\text{Var}_\theta(T_1) + \text{Var}_\theta(T_2)}\right)^2 + \frac{\text{Var}_\theta(T_1)\text{Var}_\theta(T_2)}{\text{Var}_\theta(T_1) + \text{Var}_\theta(T_2)}.\end{aligned}$$

Hence, it is minimized when $\alpha = \text{Var}_\theta(T_2)/(\text{Var}_\theta(T_1)+\text{Var}_\theta(T_2))$. If $\text{Var}_\theta(T_1)$ is very large relative to $\text{Var}_\theta(T_2)$, then α will be very small. Hence, the estimator $\alpha T_1 + (1-\alpha)T_2$ is almost similar to T_2. (d) In part (b), the variance of $\alpha T_1 + (1-\alpha)T_2$ is given by $\alpha_\theta^2\text{Var}(T_1) + (1-\alpha)_\theta^2\text{Var}(T_2) + 2\alpha(1-\alpha)\text{Cov}_\theta(T_1,T_2)$. By rearranging terms, we get

$$\begin{aligned}&\alpha^2(\text{Var}_\theta(T_1) + \text{Var}_\theta(T_2) - 2\text{Cov}_\theta(T_1,T_2))\\ &\quad - 2\alpha(\text{Var}_\theta(T_2) + \text{Cov}_\theta(T_1,T_2)) + \text{Var}_\theta(T_2).\end{aligned}$$

If $T_1 = T_2$, then $\alpha T_1 + (1-\alpha)T_2 = T_1 = T_2$ and there is nothing to do. So $P(T_1 = T_2) < 1$ is assumed. Thus, $\text{Var}_\theta(T_1)+\text{Var}_\theta(T_2) - 2\text{Cov}_\theta(T_1,T_2) = \text{Var}_\theta(T_1 - T_2) > 0$. Therefore, the variance of $\alpha T_1 + (1-\alpha)T_2$ is maximized when $\alpha = (\text{Var}_\theta(T_2)+\text{Cov}_\theta(T_1,T_2))/\text{Var}_\theta(T_1 - T_2)$. If $\text{Var}_\theta(T_1)$ is very large relative to $\text{Var}_\theta(T_2)$, then α is very small again. Hence, the linear combination estimator $\alpha T_1 + (1-\alpha)T_2$ highly depends on T_2.

6.4.1 An approximate .95-confidence interval for μ_3 is given by

$$m_3 \pm z_{\frac{1+\gamma}{2}}\frac{s_3}{\sqrt{n}} = (26.027, 151.373)$$

since $m_3 = 88.7$, $z_{.975} = 1.96$, and $s_3 = 143.0$.

6.4.2 Recall that, the variance of a random variable can be expressed in terms of the moments as $\sigma_X^2 = \mu_2 - \mu_1^2$. Hence, the method of moments estimator of the population variance is given by $\hat\sigma_X^2 = m_2 - m_1^2$. To check if this estimator is unbiased we compute

$$\begin{aligned}E\left(m_2 - m_1^2\right) &= \mu_2 - \left(\text{Var}\left(m_1\right) + E^2\left(m_1\right)\right) = \mu_2 - \left(\frac{1}{n}\left(\mu_2 - \mu_1^2\right) + \mu_1^2\right)\\ &= \left(1 - \frac{1}{n}\right)\sigma_X^2\end{aligned}$$

Hence, this estimator is not unbiased.

6.4.5 Recall from Problem 3.4.15 that the moment generating function of a $X \sim N(\mu, \sigma^2)$ is given by $m_X(s) = \exp(\mu s + \sigma^2 s^2/2)$. Then, by Theorem 3.4.3 the third moment in given by

$$m_X'''(0) = 3\sigma^2 \left(\mu + \sigma^2 s\right) e^{\mu s + \frac{1}{2}\sigma^2 s^2} + \left(\mu + \sigma^2 s\right)^3 e^{\mu s + \frac{1}{2}\sigma^2 s^2} \Big|_{s=0} = 3\sigma^2 \mu + \mu^3$$

The plug-in estimator of μ_3 is given by $\hat{\mu}_3 = 3\left(m_2 - m_1^2\right) m_1 + m_1^3$, while the method of moments estimator of μ_3 is $m_3 = \frac{1}{n} \sum x_i^3$. So these estimators are different.